$$R_\sigma(\tau, \rho) = \frac{\alpha V_c^2}{\beta(1+\alpha^2) c_l \rho} \int_0^\infty \frac{S_\omega(\omega)}{\omega} J_1\left(\frac{\omega\rho}{c_l}\right) \cos(\omega\tau)\, d\omega$$

The dispersion of quantity σ_{zz} can be found from the last formula by setting in it $\tau = 0$ and passing to the limit for $\rho \to 0$. We have

$$R_\sigma(0,0) = \frac{\alpha}{2\beta(1+\alpha^2)} \left(\frac{V_c}{c_l}\right)^2 \int_0^\infty S_\omega(\omega)\, d\omega$$

The obtained results have a simple physical interpretation. The layer of elastic medium lying over the receiver is an additional filter which transmits only those components of the external random field of pressures which satisfy the inequality $\omega > k_0 c_l$. This results in further suppression of low-frequency perturbations, as compared to the case when the receiver is directly subjected to a turbulent flow.

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VARIATIONAL PRINCIPLES OF THE THEORY OF ELASTICITY WITH VARYING INITIAL AND PERTURBED STATES

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Two variational principles of Hamilton type are presented for a nonlinear theory of elasticity, which are combined variational principles of the initial and perturbed states of elastic body motion.

Variational formulations of problems to determine the perturbed state of stress for a specified initial linear state are well-known in statics. Variational formulations have also been considered recently for the cases of a nonlinear and time-dependent initial state of stress [1 — 8]. Only quantities in the perturbed state are subjected to variation in the appropriate variational principles.

In order to avoid determining the initial state of stress in the definition of the neutral equilibrium state, varying second-order displacements were additionally introduced in the statics of an elastic body [9]. It is shown [10] that this same result is obtained in a more natural manner if an equilibrium equation is appended to the functional corresponding to the neutral equilibrium state by the method of Lagrange multipliers. On the basis of equations from analytic mechanics, a combined variational formulation has recently been given [11, 12] for the initial and perturbed states of the motion of material points system.

The ideas of the last two groups of papers are developed below.

1. Initial variational principle. Let us consider an elastic body V subjected to forces whose components remain invariant under deformation, relative to fixed curvilinear coordinate systems x_1, x_2, x_3. We assume that the strains are so small that changes in the areas and volumes during the stress calculation can be neglected. Then the geometrically nonlinear problem is described by the following equations and conditions:

$$\nabla_k [s^{kl}(\delta_l^i + \nabla_l u^i)] + X^i - \rho \ddot{u}^i = 0, \quad P \in V, \quad t_1 < t < t_2 \tag{1.1}$$

$$s^{ki} = E^{ikjl} e_{jl}, \quad e_{ik} = {}^1/_2 (\nabla_i u_k + \nabla_k u_i + \nabla_i u^l \nabla_k u_l)$$

$$s^{kj}(\delta_j^i + \nabla_j u^i)\, n_k = Q^i, \quad P \in S_1, \quad t_1 < t < t_2 \tag{1.2}$$

$$u_i = U_i, \quad P \in S_2, \quad t_1 < t < t_2 \tag{1.3}$$

$$u_i(P, t_1) = u_i'(P), \quad u_i(P, t_2) = u_i''(P), \quad P \in V \tag{1.4}$$

Here u_i are displacement vector components, e_{ik}, s_{ik} are components of the strain and stress tensors, E^{ikjl} are components of the elasticity tensor, δ_i^k is the Kronecker symbol, X^i, Q^i are components of the volume force and surface load vectors, refered to unit volume and surface of the undeformed body, respectively, U_i are given components of the displacement vector, u_i', u_i'' are given displacements for $t = t_1$ and $t = t_2$, ρ is the density of the undeformed elastic body, P is a point in the domain V occupied by the undeformed body, $S = S_1 + S_2$ is the boundary surface of the undeformed body, S_1, S_2 are parts of the surface where the external load and the displacements, respectively, are given, n_i are unit vector components of the normal to the surface S, and ∇_i is the covariant differentiation sign in the metric of the undeformed body.

The problem under consideration in nonlinear elasticity theory can be formulated by using a variational problem corresponding to the Hamilton principle as follows: find the stationary value of the functional

$$I(u) = \int_{t_1}^{t_2}\int_V \left(\frac{1}{2}\rho \dot{u}^i \dot{u}_i - \frac{1}{2} s_{t_2}^{ik} e_{ik} + X^i u_i\right) dV\, dt + \int_{t_1}^{t_2}\int_{S_1} Q^i u_i \, dS\, dt \tag{1.5}$$

under the conditions (1.3), (1.4). Equations (1.1) and the natural boundary conditions (1.2) written in displacements are the Euler-Ostrogradskii equations of the functional (1.5).

Let us assume the magnitudes for the external effect X^i, Q^i, U_i, $u_i' u_i''$ which causes a displacement u_i of the elastic body, can be represented as the sum (η is a small parameter)

$$X^i = X_0^i + \eta Y^i, \quad Q^i = Q_0^i + \eta R^i, \quad U_i = U_i^0 + \eta V_i \tag{1.6}$$

$$u_i' = u_i^{0\prime} + \eta v_i', \quad u_i'' = u_i^{0\prime\prime} + \eta v_i''$$

Correspondingly, we assume that the displacements u_i can be expanded in a power series in a small parameter

$$u_i = u_i^0 + \eta v_i + \eta^2 w_i + \ldots \qquad (1.7)$$

Substituting the quantities X^i, Q^i, U_i, u_i', u_i'', u_i expressed in terms of the relationships (1.6) and (1.7) into the functional (1.5), we obtain

$$I(u) = I_0(u_0) + \eta I_1(u_0, v) + \eta^2 I_2(u_0, v, w) + \ldots \qquad (1.8)$$

Applying the method of small parameter and using the first three terms of (1.8), we see that the variational principle

$$\delta I(u) = 0 \qquad (1.9)$$

yields the following variational principles:

$$\delta I_0(u_0) = 0, \quad \delta I_1(u_0, v) = 0, \quad \delta I_2(u_0, v, w) = 0 \qquad (1.10)$$

It is seen that the first of these principles differs from the initial variational principle (1.9) only by the notation. (Zero subscripts are ascribed to the appropriate quantities in (1.1) – (1.5)).

2. First variational principle with varying initial and perturbed motion states. Let us write the functional $I_1(u_0, v)$ in conformity with the second of the variational principles (1.10)

$$I_1(u_0, v) = \int_{t_1}^{t_2}\int_V (\rho u_0^{i\cdot} v_i^{\cdot} - s_0^{ik}\varepsilon_{ik} + X_0^i v_i + Y^i u_i^0)\, dV\, dt + \qquad (2.1)$$

$$\int_{t_1}^{t_2}\int_{S_1} (Q_0^i v_i + R^i u_i^0)\, dS\, dt$$

$$\varepsilon_{ik} = {}^1/_2(\nabla_i v_k + \nabla_k v_i + \nabla_i u_0^l \nabla_k v_l + \nabla_i v^l \nabla_k u_l^0)$$

The varying quantities u_i^0, v_i should satisfy the conditions (1.3) and (1.4), respectively, and conditions resulting from conditions (1.3) and (1.4) after taking account of (1.6) and (1.7)

$$v_i = V_i, \quad P \in S_2, \quad t_1 < t < t_2 \qquad (2.2)$$

$$v_i(P, t_1) = v_i'(P), \quad v_i(P, t_2) = v_i''(P), \quad P \in V$$

The equations of the initial state of motion (the first equation of (1.1)) and the equations

$$\nabla_k[\sigma^{kl}(\delta_l^i + \nabla_l u_0^i)] + \nabla_k(s_0^{kl}\nabla_l v^i) + X^i - \rho v^{i\cdot\cdot} = 0 \qquad (2.3)$$

$$P \in V, \quad t_1 < t < t_2$$

as well as the boundary conditions for the initial state (2.1) and the conditions

$$\sigma^{kl}(\delta_l^i + \nabla_l u_0^i) n_k + (s_0^{kl}\nabla_l v^i) n_k = R^i, \quad \sigma^{ik} = E^{ikjl}\varepsilon_{jl} \qquad (2.4)$$

$$P \in S_1, \quad t_1 < t < t_2$$

are the stationarity conditions for the functional (2.1).

Thus, equations and conditions for both the initial as well as the perturbed states originate from the functional (2.1). Therefore, the variational principle (1.10) is the combined variational formulation for these states simultaneously.

We note that an analogous variational principle has been postulated in (11, 12] for the equations of analytic mechanics. An attempt to formulate a variational principle simultaneously for the initial and perturbed states for static problems of elasticity theory is made in [13], but only by using one varying state, which does not afford the possibility of obtaining the required results.

3. Second variational principle with varying initial and perturbed motion states. We write the functional

$$I_2(u_0, v, w) = \int_{t_1}^{t_2}\int_V \Big(\rho u_0^{i\cdot} u_i^{\cdot} + \frac{1}{2}\rho v^{i\cdot} v_i^{\cdot} - s_0^{ik}\mu_{ik} - \tag{3.1}$$

$$\frac{1}{2}\sigma^{ik}\varepsilon_{ik} + X_0^i w_i + Y^i v_i\Big) dV\,dt + \int_{t_1}^{t_2}\int_{S_1} (Q_0^i w_i + R^i v_i)\,dS\,dt$$

$$\mu_{ik} = {}^1/_2\,(\nabla_i w_k + \nabla_k w_i + \nabla_i u_0^l \nabla_k w_l + \nabla_i w^l \nabla_k u_l^0 + \nabla_i v^l \nabla_k v_l)$$

The varying quantities in the functional (3.1) are u_i°, v_i, w_i, and the functions u_i°, v_i, w_i must satisfy the conditions (1.3), (1.4), (2.2), and

$$\begin{gathered} w_i = 0, \quad P \in S_2, \quad t_1 < t < t_2 \\ w_i(P, t_1) = 0, \quad w_i(P, t_2) = 0, \quad P \in V \end{gathered} \tag{3.2}$$

The stationarity conditions for the functional (3.1) are the three groups of equations: the equations of the initial state of motion (the first equation in (1.1)), the equation of the perturbed state of motion (2.3) and the equations

$$\begin{gathered} \nabla_k[\tau^{kl}(\delta_l^i + \nabla_l u_0^i)] + \nabla_k(s_0^{kl}\nabla_l w^i) + \nabla_k(\sigma^{kl}\nabla_l v^i) - \rho w^{i\cdot\cdot} = 0 \\ P \in V, \quad t_1 < t < t_2 \end{gathered} \tag{3.3}$$

as well as the appropriate boundary conditions (1.2), (2.4) and

$$\begin{gathered} \tau^{kl}(\delta_l^i + \nabla_l u_0^i) n_k + (s_0^{kl}\nabla_l w^i) n_k + (\sigma^{kl}\nabla_l v^i) n_k = 0 \\ \tau^{ik} = E^{ikjl}\mu_{jl}, \quad P \in S_1, \quad t_1 < t < t_2 \end{gathered} \tag{3.4}$$

We now consider all perturbations of external effects to be zero, i.e. we consider the problem of stability of an elastic body. If it is assumed in this case that the displacements of the initial state satisfy the first equation in (1.1) and the initial and boundary conditions (1.2) and (1.4), then a well-known form without the varying displacements w_i

$$I_2'(v) = \int_{t_1}^{t_2}\int_V \Big(\frac{1}{2}\rho v^{i\cdot} v_i^{\cdot} - s_0^{ik}\nabla_i v^l \nabla_k v_l - \frac{1}{2}\sigma^{ik}\varepsilon_{ik}\Big)\,dV\,dt$$

can be given to the functional (3.1).

Another possibility of transforming the functional (3.1) exists, where the additional conditions (3.2) — (3.4) are used in place of the additional conditions (1.1) — (1.4). An appropriate functional for linearized problems is presented in [9, 10]. This transformation yields no essential simplification of the functional (3.1) in the general nonlinear case.

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SOLUTION OF A HOMOGENEOUS BOUNDARY VALUE PROBLEM FOR THE SECTOR OF A TOROIDAL SHELL SEGMENT

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Membrane forces in the segment of a thin toroidal shell loaded by an edge bending load are determined from the particular solution of the fundamental differential equation. Taking account of the asymptotic approximation of the special function in whose terms the particular solution is expressed, it is shown in [1] that the particular solution for a thin toroidal shell agrees with the membrane solution. In the general case, the tensile forces in a shell not closed in two coordinates are determined by membrane theory; the membrane state of stress is determined